DRAFT of

Folding
Linkages, Origami, and Polyhedra

NOT FOR DISTRIBUTION

Joseph O'Rourke[1]

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[1] Department of Computer Science, Smith College, Northampton, MA 01063, USA. `orourke@cs.smith.edu`.

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These are excerpts from Part II of a book draft. The book is aimed at high-school students, and should be readable at the middle-school level. (Part I is on linkages, Part III is on polyhedra.) The title of the book is tentative.

Part II

Origami

Chapter 4

Flat Vertex Folds

Theorem 1 *A vertex in a flat folding has even degree.*

Theorem 2 (Maekawa-Justin) *If M mountain creases and V valley creases meet at a vertex of a flat folding, then M and V differ by 2: either $M{=}V+2$ or $V{=}M+2$.*

Theorem 3 (Local Min) *In any flat folding, any wedge whose angle is a local min must be delimited by one mountain and one valley fold.*

Theorem 4 (Kawasaki-Justin) *A set of an even number of creases meeting at a vertex folds flat if, and only if, the alternating sum of the determined wedge angles is zero:*

$$\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots - \theta_n = 0^\circ \ .$$

Consider the 6-crease example in Figure 4.1(a), with six wedge angles

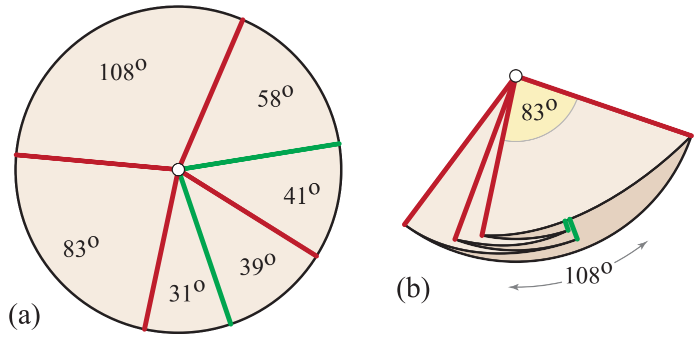

Figure 4.1: Illustration of Kawasaki Theorem 4: 31°+41°+108° = 39°+58°+83°.

$$31^\circ + 39^\circ + 41^\circ + 58^\circ + 108^\circ + 83^\circ = 360^\circ \ .$$

Their alternating sum is indeed zero:

$$31^\circ + 41^\circ + 108^\circ = 180^\circ = 39^\circ + 58^\circ + 83^\circ \ ,$$

so

$$31^\circ - 39^\circ + 41^\circ - 58^\circ + 108^\circ - 83^\circ = 0^\circ \ .$$

The flat folding guaranteed by the theorem is shown in (b) of the figure.

Chapter 5

Fold And Cut

Theorem 5 *Any straight-line drawing (one composed of straight* segments*) on a sheet of paper may be folded flat so that one straight scissors cut completely through the folding cuts all the segments of the drawing and nothing else.*

Figure 5.1(a) shows the magnification and reduction for the boundary in a representation of the letter `A`, resulting in the straight skeleton shown in (b).

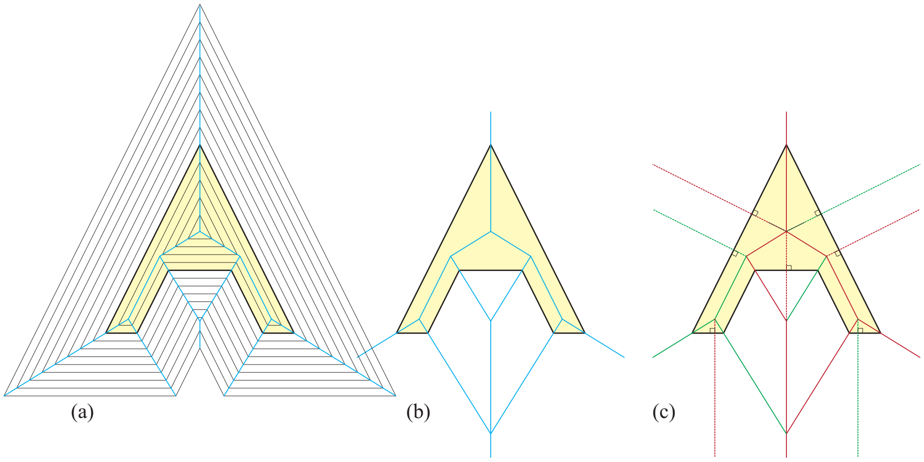

Figure 5.1: (a) Growing and shrinking the shape. (b) Straight skeleton. (c) Perpendiculars added, and mountain/valley folds indicated.

fold-and cut:	1D segments	on 2D paper	fold to 1D line
polyhedron flattening:	2D faces	in 3D space	fold to 2D plane

Chapter 6

The Shopping Bag Theorem

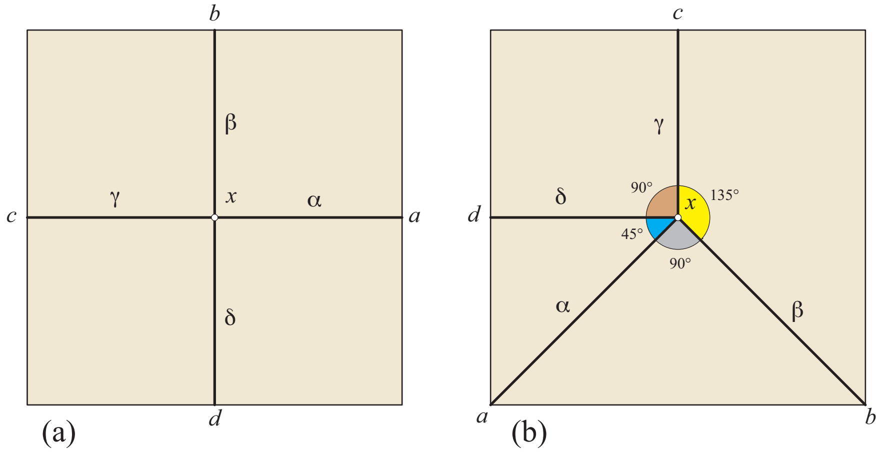

Figure 6.1: (a) Creases meeting orthogonally. (b) Creases meeting as in the Miura fold.

Lemma 6 (Degree 4) *Let four creases meet at vertex x, with dihedral angles $\alpha, \beta, \gamma, \delta$ along segments ax, bx, cx, dx respectively, in either of the two configurations illustrated in Figure 6.1. Then the dihedral angles along opposite creases are equal: $\alpha = \gamma$ and $\beta = \delta$. Moreover, if the four creases meet perpendicularly (a), then if α is more than 0° and less than 180°, then either $\beta = 0^\circ$ or $\beta = 180^\circ$.*

Lemma 7 (Degree 3) *If exactly three creases are incident to a vertex x, no two of which lie on the same line, then all their dihedral angles are 180° (i.e., they are not creases), so x is not a true vertex—the neighborhood of x is flat.*

Theorem 8 (Shopping Bag) *If the faces of a tall shopping bag are rigid, then the bag may be either fully opened (as illustrated), or fully collapsed flat. It has* *no other configurations. In particular, the bag is rigid in either the opened or flattened configuration.*

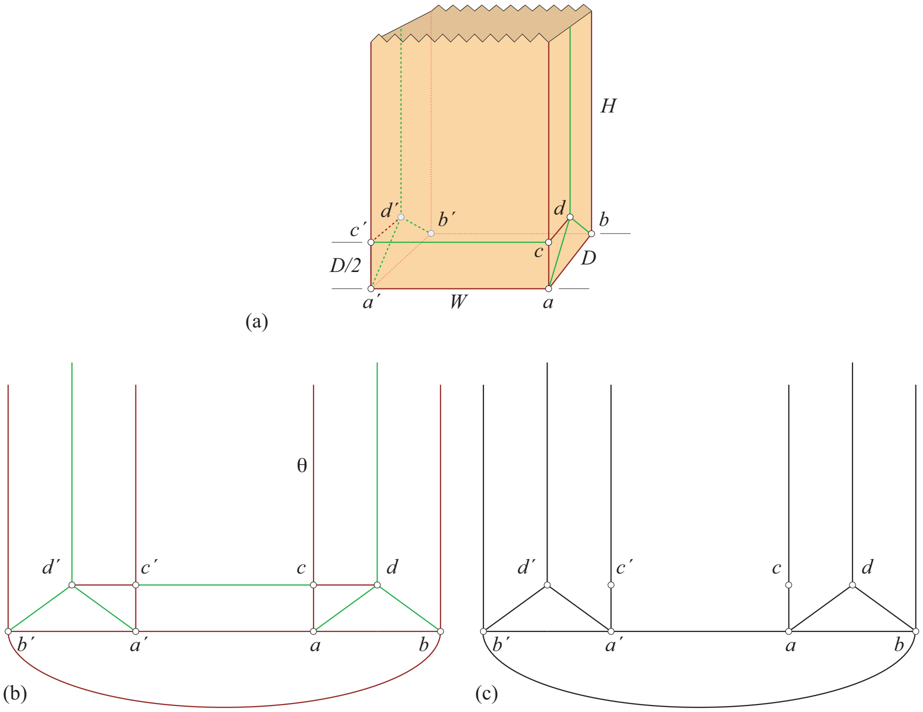

Figure 6.2: (a) Model of a "tall" grocery shopping bag. (b) Network of creases between vertices. (c) Network when $0 < \theta < 180°$.

We say that any bag with $H > \frac{1}{2}D$ is a *tall shopping bag*.